REPRESENT A NATURAL NUMBER AS THE SUM OF PALINDROMES IN VARIOUS BASES

YU GAO

Abstract. It is shown that the set of palindromes is an additive basis for the natural numbers in any base. Specifically, we prove that every natural number can be expressed as the sum of $O(d)$ palindromes in base d.

1. Statement of result

Let $\mathbb{N} := \{0, 1, 2, \ldots\}$ denote the set of natural numbers. For any integer $d \geqslant 2$, every number $n \in \mathbb{N}$ has a unique representation of the form

$$n = \sum_{j=0}^{l-1} 10^j \delta_j, \tag{1.1}$$

where each digit δ_j belongs to the digit set

$$\mathcal{D} := \{0, 1, 2, \ldots, d-1\},$$

and the leading digit δ_{l-1} is nonzero whenever $l \geqslant 2$. We use

$$n = \boxed{\delta_{l-1}} \boxed{\cdots} \boxed{\delta_1} \boxed{\delta_0}$$

represents the relation (1.1). The integer n is said to be a *palindrome* if its digits satisfy the symmetry condition

$$\delta_j = \delta_{l-1-j} \qquad (0 \leqslant j < l).$$

Denoting by P_d the collection of all palindromes in $\mathbb{N}$, we are to show that P_d is an additive basis for $\mathbb{N}$ and for any $d \geqslant 2$.

Theorem 1.1. *Every natural number is the sum of $O(d)$ palindromes in base d for integer $d \geqslant 2$.*

---

[1] proved the decimal case using a different approach. Our construction is simpler and more generalized but it uses more palindromes than [1].

2. The proof

2.1. **Notation.** We denote by $P_{d,l,k}$ the set of integers $\{c(d^{l-1}+d^k)|0 \leqslant c < d, 0 \leqslant k < l-1\}$. Next we denote by $\mathbb{N}_{l,k}$ the set of integers $\{xd^k|d^{l-1-k} \leqslant x < d^{l-k}\}$. In other words, $\mathbb{N}_{l,k}$ is the set of natural numbers n that have a d base expansion of the form

$$n = \boxed{\delta_{l-1}} \cdots \boxed{\delta_k} \boxed{0} \cdots \boxed{0}$$

with $\delta_{l-1} \neq 0$.

2.2. **Inductive passage from $\mathbb{N}_{l,k}$ to $\mathbb{N}_{l-1,k+1}$.**

LEMMA 2.1. *Every number $n \in \mathbb{N}_{l,k}$ is the sum of at most $2d$ numbers given by Algorithm 1 from $P_{d,l,k}$ and some number $n' \in \mathbb{N}_{l-1,k+1}$ for $l \geqslant k + 6$.*

Algorithm 1 Inductive passage from $\mathbb{N}_{l,k}$ to $\mathbb{N}_{l-1,k+1}$.

Require: an integer $n \in \mathbb{N}_{l,k}$.
Ensure: a multiset S_k and the modified n.
 1: $S_k \leftarrow \varnothing$
 2: $x \leftarrow (d-1)d^{l-2} + (d-1)d^k$
 3: **while** $n \geqslant d^{l-1} + x$ **do**
 4:      $n \leftarrow n - x$
 5:      $S_k \leftarrow S_k \cup \{x\}$
 6: **end while**
 7: $y \leftarrow (d-1)d^{l-3} + (d-1)d^k$
 8: **while** $n \geqslant d^{l-1}$ **do**
 9:      $n \leftarrow n - y$
10:      $S_k \leftarrow S_k \cup \{y\}$
11: **end while**
12: $c \leftarrow \delta_k(n)$
13: $z \leftarrow cd^{l-4} + cd^k$
14: $S_k \leftarrow S_k \cup \{z\}$
15: $n \leftarrow n - z$
16: **return** S_k and n

Proof. Since $\mathbb{N}_{l,k} \subset \{x|d^{l-1} \leqslant x < d^l\}$, we have

LEMMA 2.2. *The first loop in Algorithm 1 executes for at most $d - 1$ times.*

Proof. If the first loop executes for at least d times, then $n \geqslant d^{l-1} + dx \geqslant d^l$. $\square$

It's clear that $d^{l-1} \leqslant n < d^{l-1} + x$ after the first loop. Similarly, we have

LEMMA 2.3. *The second loop in Algorithm 1 executes for at most d times.*

Proof. If the second loop executes for at least $d + 1$ times, then $n \geqslant d^{l-1} + dy \geqslant d^{l-1} + x$. $\square$

Thus $|S_k| \leqslant 2d$ for the returned $|S_k|$. To complete the proof, it suffices to show that the returned n by Algorithm 1 is in $N_{l-1,k+1}$. $\delta_k(n) = 0$ is satisfied by the choice of c. By the end condition of the second loop,

$$
\begin{aligned}
d^{l-1} > n &\geqslant d^{l-1} - y - z \\
&\geqslant d^{l-1} - ((d-1)d^{l-3} + (d-1)d^k) - ((d-1)d^{l-4} + (d-1)d^k) \\
&= d^k(dd^{l-k-2} - ((d-1)d^{l-k-3} + (d-1)d^{l-k-4} + d + (d-2))) \\
&\geqslant d^k((d-1)d^{l-k-2}) \\
&= (d-1)d^{l-2} \\
&\geqslant d^{l-2}.
\end{aligned}
$$

$\square$

2.3. **Pseudo-Theorem 1.**

LEMMA 2.4. *If n is a natural number with at most k nonzero d base digits, then n is the sum of $2k$ palindromes in base d.*

Proof. Let f be the function as following:

$$
f(x) := \begin{cases} 0 & \text{if } x = 0; \\ 1 & \text{if } x = 1; \\ d - x + 1 & \text{if } x = 1, 2, \ldots, d-1. \end{cases} \tag{2.1}
$$

So

$$
\begin{aligned}
n &= \sum_k d^k \delta_k \\
&= \sum_k f(\delta_k) + (d^k \delta_k - f(\delta_k)).
\end{aligned}
$$

The proof is completed by the observation that $f(\delta_k)$ and $d^k \delta_k - f(\delta_k)$ are both palindromes. $\square$

LEMMA 2.5. *(Pseudo-Theorem 1)Every number is the sum of $O(d)$ pseudo-palindromes in base d for $d \geqslant 2$.*

Proof. A pseudo-palindrome here is a palindrome with possibly leading zeros. For any number $n \in \mathbb{N}_{l,0}$, using Algorithm 1 repeatedly, we have a family of sets $\{S_k | l - k \geqslant k + 6\}$. S_k consists at most $(d-1)$ numbers in $P_{l-k-1,k}$(denoted by $s_{k,j}, 0 \leqslant j < d$), d numbers in $P_{l-k-2,k}$(denoted by $t_{k,j}, 0 \leqslant j \leqslant d$), and 1 number in $P_{l-k-3,k}$(denoted by r_k). (We add zeros in case of $|S_k| < 2d$.) Notice that $s_{k,j}$

only have two nonzero d base digits, namely δ_{l-k-2} and δ_k. Considering the form of d-base expansions,

$$\sum_k s_{k,j} = \boxed{\delta_{l-2}(s_{0,j})}\ \boxed{\delta_{l-3}(s_{1,j})}\ \boxed{\ \ldots\ }\ \boxed{\delta_1(s_{1,j})}\ \boxed{\delta_0(s_{0,j})}$$

is a pseudo-palindrome; so are $\sum_k t_{k,j}$ and $\sum_k r_k$. Thus n is the sum of $2d$ pseudo-palindromes and a number m in $\mathbb{N}_{l',l''}$ for $l' - l'' \leqslant 5$. By Lemma 2.4, m is also a sum of at most 10 palindromes. $\qquad\square$

2.4. **Proof of Theorem 1.1.** The only "bug" in Pseudo-Theorem 1(Lemma 2.5) is that the $2d$ pseudo-palindromes may have leading zeros when $0 \in S_0$(i.e. when some of $\delta_{l-2}(s_{0,j})$, $\delta_{l-3}(t_{0,j})$ and $\delta_{l-4}(r_j)$ are equal to zero). So we must reduce a natural number n to some number with $0 \notin S_0$ in advance.

LEMMA 2.6. *Let* $n \in \mathbb{N}_{l,0}$ *for* $l \geqslant 8$. *Then n is the sum of* $O(d)$ *palindromes produced by Algorithm 2 and an natural number* $f(n)$ *of the form* $(d-1)d^{l-2} + (d-1)d^{l-3} + (d-1)d^{l-4} + m$ *for* $d^2 - 2d \leqslant m < d^{l-4}$.

Algorithm 2 Reduce to a proper initial value.

Require: an integer $n \in \mathbb{N}_{l,0}$.
Ensure: a multiset T and $f(n)$.
 1: $T \leftarrow \varnothing$
 2: $x \leftarrow (d-1)d^{l-2} + (d-1)$
 3: **while** $n \geqslant d^{l-1} + x$ **do**
 4:     $n \leftarrow n - x$
 5:     $T \leftarrow T \cup \{x\}$
 6: **end while**
 7: $y \leftarrow (d-1)d^{l-3} + (d-1)$
 8: **while** $n \geqslant d^{l-1} + y$ **do**
 9:     $n \leftarrow n - y$
10:     $T \leftarrow T \cup \{y\}$
11: **end while**
12: $z \leftarrow (d-1)d^{l-4} + (d-1)$
13: **while** $n \geqslant d^{l-1} + z$ **do**
14:     $n \leftarrow n - z$
15:     $T \leftarrow T \cup \{z\}$
16: **end while**
17: $w \leftarrow (d-1)d^{l-5} + (d-1)$
18: **while** $n \geqslant d^{l-1}$ **do**
19:     $n \leftarrow n - w$
20:     $T \leftarrow T \cup \{w\}$
21: **end while**
22: **return** T and n(as $f(n)$)

Proof. As the analysis of Algorithm 1, we can easily show that $|T| = O(d)$ and $d^{l-1} - w \leqslant n < d^{l-1}$ after Algorithm 2.

$$
\begin{aligned}
m &= n - (d-1)d^{l-2} - (d-1)d^{l-3} - (d-1)d^{l-4} \\
&\geqslant d^{l-4} - w \\
&= d^{l-5} - (d-1) \\
&\geqslant d^2 - 2d.
\end{aligned}
$$

$\square$

We are now ready to give a proof to Theorem 1.1. If $n \leqslant d^8 + 1$, n is the sum of at most 16 palindromes by Lemma 2.4. Otherwise we use Algorithm 1 on $f(n)$ produced by Algorithm 2. In Algorithm 1, both the first and the second loop executes for $(d-1)$ times for $d \geqslant 3$, when the second loop executes for one more time for $d = 2$. (The inequality $d^2 - 2d \leqslant m < d^{l-4}$ provides that we don't need to care about the trailing digits of x, y and z in Algorithm 1.) In the case of $d \geqslant 3$, we modifies S by replacing one of the y produced by the second loop of Algorithm 1 with two nonzero palindromes $(d-2)d^{l-3} + (d-2)d^k$ and $d^{l-3} + d^k$. If $c = 0$, we just decrease n by 1(1 is a palindrome) before simulating Algorithm 1. Combining the pitches we have $0 \notin S_0$ and the pseudo-palindromes become genuine palindromes.

References

[1] W. D. Banks, "Every natural number is the sum of forty-nine palindromes," *arXiv:1508.04721*.

ACM honored class, Shanghai Jiao Tong University, Shanghai, China.
E-mail address: xyz2606@sjtu.edu.cn